

NEW CONDITIONS FOR CENTRAL LIMIT THEOREMS

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1. Introduction

A general formulation of the central limit problem for sums of independent random variables is the following (see Loève [1], p. 291).

Let

$$S_n = \sum_k X_{nk}$$

where $k=1, \dots, k_n$, $k_n \rightarrow \infty$ as $n \rightarrow \infty$, and for each n $\{X_{nk}\}$ are independent random variables with probability distribution functions F_{nk} and $EX_{nk} = 0$. Let $\{F_n\}$ be the distribution functions of $\{S_n\}$ and let $\phi(x)$ be the distribution function of a Normal random variable with zero-mean and variance σ^2 . Under these conditions it is possible to show the following.

Theorem 1.1: Let $\max_k \text{Var } X_{nk} \rightarrow 0$ and $\sum_k \text{Var } X_{nk} \rightarrow \sigma^2 < \infty$ where σ^2 is a positive constant. The sums S_n are asymptotically Normal (i.e., $F_n(x) \rightarrow \phi(x)$) if and only if for every $\epsilon > 0$

$$(1.1) \quad g_n(\epsilon) = \sum_k \int_{|x| \geq \epsilon} x^2 dF_{nk} \rightarrow 0.$$

Except in special cases, the application of condition 1.1 is difficult because of the integrals involved. By assuming the existence of

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fourth-order moments, we are able to prove new necessary and sufficient conditions for both Normal and Poisson convergence which involve only moments. The proof of the theorem makes use of a characterization of the Normal distribution among infinitely divisible (I.D.) laws which was perhaps first recognized by Borges [3] and later independently by the author [4].

2. Normal Convergence

Theorem 2.1: Let $E|S_n|^{(4+\delta)}$ be uniformly bounded for some $\delta > 0$. Let $\max_k \text{Var } X_{nk} \rightarrow 0$, $\sum_k \text{Var } X_{nk} \rightarrow \sigma^2 < \infty$ where σ^2 is a positive constant. Then S_n is asymptotically Normal if and only if

$$(2.1) \quad ES_n^4 - 3\{ES_n^2\}^2 \rightarrow 0.$$

Proof: The asymptotic normality of S_n implies condition 2.1 by the moment convergence theorem (see Loeve [1], p. 184) and the fact that for a zero-mean Normal random variable S , $ES^4 - 3\{ES^2\}^2 = 0$.

To prove the converse it is sufficient to show that every convergent subsequence $\{F_{n'}\}$ of $\{F_n\}$ converges to $\phi(x)$ (see Feller [2], p. 261).

Let F be the limit of $F_{n'}$, then F is an infinitely divisible law with characteristic function $f(u)$ such that (see Loève [1], p. 293),

$$(2.2) \quad \log f(u) = \int (e^{iux} - 1 - iux) \frac{1}{x^2} dK(x)$$

where $K(x)$ is monotone increasing and of bounded variation, $K(-\infty) = 0$, and $K(\infty) = \sigma^2 < \infty$. The integrand is defined by continuity

at the origin. It is known that $F(x)$ is a Normal law if and only if $K(x)$ increases only at $x = 0$.

Since $E|S_n|^{(4+\delta)}$ are uniformly bounded, all moments of S_n , of order 4 or less converge to those of F . Since $ES_n^4 - 3\{ES_n^2\}^2 \rightarrow 0$,

$$\begin{aligned} 0 &= \int x^4 dF(x) - 3 \int x^2 dF(x)^2 \\ &= \frac{\partial^4}{\partial u^4} \log f(u) \Big|_{u=0} \\ (2.3) \quad &= \int x^2 dK(x). \end{aligned}$$

This last equation is obtained by differentiating the RHS of Eq. (2.2) under the integral sign. This is justified in the following way. That

$$(2.4) \quad -\frac{d^2}{du^2} \log f(u) = \int e^{iux} dK(x)$$

is shown by Loeve [1], p. 293. Thus the L.H.S. of Eq. (2.4) is a characteristic function. Since this characteristic function is twice differentiable, its second derivative is given by (Loève [1], p. 200)

$$\frac{d^4}{du^4} \log f(u) = \int x^2 e^{iux} dK(x)$$

and Eq. (2.3) follows. Thus $K(x)$ increases only at $x = 0$.

Finally, since $\sum_k \text{Var } X_{nk} \rightarrow \sigma^2$ we have shown that $F_n \rightarrow \phi(x)$ and the proof is complete.

Condition 2.1 becomes even simpler when we note that

$$ES_n^4 - 3\{ES_n^2\}^2 = \sum_k \left[EX_{nk}^4 - 3\{EX_{nk}^2\}^2 \right].$$

The L.H.S. is usually called the fourth cumulant of S_n . This identity says that the fourth cumulant of a sum of independent random variables equals the sum of the fourth cumulants.

If the distributions F_n are known to be infinitely divisible (I.D.), then moments higher than 4 are not required.

Theorem 2.2: If F_n are I.D., then $F_n(x) \rightarrow \phi(x)$ if and only if $ES_n^2 \rightarrow \sigma^2$ and

$$ES_n^4 - 3\{ES_n^2\}^2 \rightarrow 0.$$

Proof: The characteristic functions $f_n(u)$ are given by

$$\log f_n(u) = \int (e^{iux} - 1 - iux) \frac{1}{x^2} dK_n(x).$$

For any $\epsilon > 0$

$$\int_{|x| > \epsilon} dK_n(x) \leq \int \frac{x^2}{\epsilon^2} dK_n(x) \rightarrow 0$$

as $n \rightarrow \infty$. Thus $K_n(x)$ converges to a step function at the origin of size σ^2 .

The converse is obtained from the moment convergence theorem.

3. Poisson Convergence

The key feature of the results above is that the fourth cumulant (i.e., $ES_n^4 - 3\{ES_n^2\}^2$) corresponds to $\int x^2 dK(x)$ and is a good test for a jump of $K(x)$ at the origin. However, this method can be used to test for jumps at other points also. For example, the equation

$$\int (x-1)^2 dK(x) = 0$$

implies that $K(x)$ can only have a jump at $x = 1$. This leads to the following results. First we note that

$$\begin{aligned} \int (x-1)^2 dK_n(x) &= ES_n^4 - 3\{ES_n^2\}^2 - 2ES_n^3 + ES_n^2 \\ &= \sum_k \left[EX_{nk}^4 - 3\{EX_{nk}^2\}^2 - 2EX_{nk}^3 + EX_{nk}^2 \right]. \end{aligned}$$

Theorem 3.1: Let $E|S_n|^{(4+\delta)}$ be uniformly bounded for some $\delta > 0$. Then S_n is asymptotically Poisson (i.e., $\log f_n(u) \rightarrow [\sigma^2(e^{iu}-1)-i\sigma^2u]$) if and only if

$$ES_n^4 - 3\{ES_n^2\}^2 - 2ES_n^3 + ES_n^2 \rightarrow 0.$$

In a similar way we obtain:

Theorem 3.2: If F_n are I.D., then F_n is asymptotically Poisson if and only if $ES_n^2 \rightarrow \sigma^2$ and

$$ES_n^4 - 3\{ES_n^2\}^2 - 2ES_n^3 + ES_n^2 \rightarrow 0.$$

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